ON THE DISTRIBUTION OF QUADRATIC NON-RESIDUES AND THE EUCLIDEAN ALGORITHM IN REAL QUADRATIC FIELDS. II

BY

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1. **Introduction.** It is the purpose of the paper to establish the following result(1):

For a prime $p \equiv 17$ (mod 24), *there is no E.A. in* $R(p^{1/2})$ *except possibly for* $p = 17, 41, 89, 113$, *and* 137.

It is known that for $p = 17, 41$ the field is Euclidean. The only doubtful cases are $p = 89, 113$, and 137.

The method used is a refinement of the previous paper, by introducing Euler's summation formula and a new estimation of character sum.

The numbering of theorems and lemmas continues from I, and the notations of I are preserved, for example p denotes a prime, $[\xi]$ denotes the integral part of ξ and $q_1 < q_2 < q_3$ denote the least three primes which are quadratic non-residues, mod p.

2. **Lemmas involving primes.**

LEMMA 17. *For* $11 \leqq x \leqq 10^6$,

$$li\ x - li\ x^{1/2} < \pi(x) < li\ x$$

(Rosser, Theorems 1 and 3).

LEMMA 18. *For* $x \geqq 400$, $\vartheta(x) > (1-0.139)x$.

Proof. For $x < 10^6$ use Theorem 7(2) of Rosser, and for $x > 10^6$, use (10)(3) of Rosser.

LEMMA 19. *For* $401 \leqq a \leqq b \leqq a^2$,

$$\sum_{a \leqq q \leqq b} \frac{1}{q^2} < \frac{1.2142}{a \log a}$$

where q runs over primes.

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(1) Editor's footnote. The proof given here is entirely different from that given in L. Rédei, *Zur Frage des Euklidischen Algorithmus in quadratischen Zahlkörpern*, Math. Ann. vol. 118 (1942) pp. 588–608. In that paper Rédei proved that the Euclidean algorithm does not exist in quadratic fields $R(\mu^{1/2})$ if $\mu > 41$ has the form $24n+17$. The paper is unknown to the author and arrived only recently in this country.

(2) For $0 < x \leqq 10^6$, $x - 2.78x^{1/2} < \vartheta(x)$.

(3) For $x > e^{13.8}$, $\vartheta(x) > (1-0.0393)x$.

Proof. By Lemma 1 (of I) and Lemma 18,

$$\sum_{a\leqq q\leqq b}\frac{1}{q^2}\leqq\sum_{a\leqq n\leqq b}\frac{\vartheta(n)-\vartheta(n-1)}{n^2\log n}$$

$$=\sum_{a\leqq n\leqq b}\vartheta(n)\left(\frac{1}{n^2\log n}-\frac{1}{(n+1)^2\log(n+1)}\right)$$

$$+\frac{\vartheta(b)}{(b+1)^2\log(b+1)}-\frac{\vartheta(a-1)}{a^2\log a}$$

$$<1.0376\left(\sum_{a\leqq n\leqq b}n\left(\frac{1}{n^2\log n}-\frac{1}{(n+1)^2\log(n+1)}\right)\right.$$

$$\left.+\frac{b}{(b+1)^2\log(b+1)}\right)-\frac{(1-0.139)(a-1)}{a^2\log a}$$

$$=1.0376\sum_{a\leqq n\leqq b}\frac{1}{n^2\log n}+\frac{0.1766(a-1)}{a^2\log a}$$

$$<\frac{1.0376}{\log a}\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{a^2}\right)+\frac{0.1766}{a\log a}$$

$$<\frac{1.2142}{a\log a}.$$

3. Some results concerning trigonometrical sums.

LEMMA 20. *If A is an integer not less than 1, $0<y\leqq\pi/2A$, and $|x|\leqq 1$, we have*

$$\left|\sum_{n=1}^{A}(-1)^n\frac{\sin xyn}{\sin yn}\right|\leqq 1.$$

Proof. Without loss of generality we assume $x>0$. The function $\sin xz/\sin z$ increases monotonically for $0<z\leqq\pi/2$. In fact

$$\frac{d}{dz}\frac{\sin xz}{\sin z}=\frac{x\sin z\cos xz-\cos z\sin xz}{\sin^2 z}\geqq 0,$$

since the numerator of the expression vanishes at $z=0$ and its derivative is $(1-x^2)\sin z\sin xz\geqq 0$ for $0<z\leqq\pi/2$.

Thus the sum is an alternating series with increasing absolute values of terms. Since $0<x\leqq 1$ and $0<yn\leqq\pi/2$, we have

$$0<\sin xyn/\sin yn\leqq 1;$$

the lemma follows immediately.

LEMMA 21. *Let A be an integer not less than* 1 *and m be a positive integer not greater than $p/2A$ and let*

$$S_m = S_m(A) = 2 \sum_{x=1}^{[(p-1)/2m]} \left(\frac{\sin \pi(A+1)mx/p}{\sin \pi mx/p} \right)^2.$$

Then

$$| S_m - p(A+1)/m + (A+1)^2 | < 3A + 1$$

and

$$S_1 = p(A+1) - (A+1)^2.$$

Proof. For the sake of brevity, we write

$$e(x) = e^{2\pi i x},$$

and we use $\mathcal{R}x$ to denote the real part of x.

We have

$$\begin{aligned} S_m &= 2 \sum_{x=1}^{[(p-1)/2m]} \sum_{a=0}^{A} \sum_{n=-a}^{a} e(mnx/p) \\ &= 2\left[\frac{p-1}{2m}\right](A+1) + 4 \sum_{a=1}^{A} \sum_{n=1}^{a} \\ &\qquad \cdot \mathcal{R}\left(\frac{2e((mn/p)([(p-1)/2m] + 1/2)) - e(mn/2p) + e(-mn/2p)}{2(e(mn/2p) - e(-mn/2p))} \right) \\ &= 2\left[\frac{p-1}{2m}\right](A+1) - A(A+1) \\ &\qquad + 4 \sum_{a=1}^{A} \sum_{n=1}^{a} \mathcal{R} \frac{e((mn/p)([(p-1)/2m] + 1/2))}{e(mn/2p) - e(-mn/2p)} \\ &= 2\left[\frac{p-1}{2m}\right](A+1) - A(A+1) \\ &\qquad + 2 \sum_{a=1}^{A} \sum_{n=1}^{a} \frac{\sin([(p-1)/2m] + 1/2)2\pi mn/p}{\sin \pi mn/p} \\ &= 2\left[\frac{p-1}{2m}\right](A+1) - A(A+1) \\ &\qquad + 2 \sum_{a=1}^{A} \sum_{n=1}^{a} (-1)^n \frac{\sin([(p-1)/2m] - p/2m + 1/2)2\pi mn/p}{\sin \pi mn/p} \\ &= 2\left[\frac{p-1}{2m}\right](A+1) - A(A+1) + 2A\theta, \end{aligned}$$

where $|\theta| \leqq 1$, by Lemma 20, since $m \leqq p/2A \leqq p/2a$ and

$$-1 + 1/2m \leqq [(p-1)/2m] - (p-1)/2m \leqq 0.$$

Since

$$-2(A+1) \leqq 2[(p-1)/2m](A+1) - p(A+1)/m \leqq -(A+1)/m < 0$$

we have

$$-4A - 2 < S_m - p(A+1)/m + A(A+1) < 2A.$$

Hence the inequality of the lemma follows.

In particular, when $m=1$, we have $\theta=0$. Hence the equality follows.

LEMMA 22. *For* $\alpha \geqq 0$,

$$\sum_{n=1}^{\infty} \frac{\sin^2 \pi n\alpha}{n^2} = \frac{\pi^2}{2}(\{\alpha\} - \{\alpha\}^2),$$

where $\{\alpha\} = \alpha - [\alpha]$.

(See, for example, Whittaker-Watson, *Modern Analysis*, 4th ed., Example 3, p. 163.)

LEMMA 23. *Let* $A \geqq 0$, $m > 0$, *then*

$$-\frac{4p}{\pi^2 m} - \frac{8}{\pi^2} < S_m - \frac{p^2}{m^2}\left(\left\{\frac{(A+1)m}{p}\right\} - \left\{\frac{(A+1)m}{p}\right\}^2\right) < \frac{5p}{6m}.$$

Proof. (1) Using $\sin x \leqq x$ for $0 \leqq x \leqq \pi/2$, we have

$$\begin{aligned} S_m &> 2 \sum_{x=1}^{[(p-1)/2m]} \frac{\sin^2 \pi(A+1)mx/p}{(\pi m x/p)^2} \\ &> \frac{2p^2}{\pi^2 m^2}\left(\sum_{x=1}^{\infty} \frac{\sin^2 \pi(A+1)mx/p}{x^2} - \sum_{x=[(p-1)/2m]+1}^{\infty} \frac{1}{x^2}\right) \\ &> \frac{2p^2}{\pi^2 m^2}\left(\sum_{x=1}^{\infty} \frac{\sin^2 \pi(A+1)mx/p}{x^2} - \frac{2m}{p} - \frac{4m^2}{p^2}\right), \end{aligned}$$

since $[(p-1)/2m]+1 > (p-1)/2m - (2m-1)/2m + 1 = p/2m$ and

$$\sum_{x=[(p-1)/2m]+1}^{\infty} \frac{1}{x^2} < \int_{p/2m}^{\infty} \frac{dx}{x^2} + \frac{4m^2}{p^2} = \frac{2m}{p} + \frac{4m^2}{p^2}.$$

By Lemma 22,

$$S_m > \frac{p^2}{m^2}\left(\left\{\frac{(A+1)m}{p}\right\} - \left\{\frac{(A+1)m}{p}\right\}^2\right) - \frac{4p}{\pi^2 m} - \frac{8}{\pi^2}.$$

(2) For $0 \leqq x \leqq 0.44$, we have

$$(1 - x)^{-2} \leqq 1 + 5x$$

and for $0<x\leqq\pi/2$, we have

$$\sin x > x - x^3/6.$$

Therefore, for $1<x\leqq(p-1)/2m$,

$$\frac{1}{(\sin \pi mx/p)^2} < \frac{1}{(\pi^2m^2x^2/p^2)(1 - \pi^2m^2x^2/6p^2)^2} < \frac{p^2}{\pi^2m^2x^2}\left(1 + \frac{5\pi^2m^2x^2}{6p^2}\right)$$

since

$$\pi^2m^2x^2/6p^2 < \pi^2/24 < 0.44.$$

Hence, by Lemma 22,

$$\begin{aligned} S_m &< 2\sum_{x=1}^{[(p-1)/2m]} \frac{p^2}{\pi^2m^2}\frac{\sin^2 \pi(A+1)mx/p}{x^2}\left(1 + \frac{5\pi^2m^2x^2}{6p^2}\right) \\ &< \frac{2p^2}{\pi^2m^2}\left(\sum_{x=1}^{\infty} \frac{\sin^2 \pi(A+1)mx/p}{x^2} + \sum_{x=1}^{[(p-1)/2m]} \frac{5\pi^2m^2}{6p^2}\right) \\ &< \frac{p^2}{m^2}\left(\left\{\frac{(A+1)m}{p}\right\} - \left\{\frac{(A+1)m}{p}\right\}^2\right) + \frac{5p}{6m}\,. \end{aligned}$$

4. Result concerning character sums.

LEMMA 24. *Let $p>10^6$ and let*

$$N = N(A) = 2\sum_{x=1}^{(p-1)/2}{}' \left(\frac{\sin \pi(A+1)x/p}{\sin \pi x/p}\right)^2,$$

where the prime denotes that the sum runs over quadratic non-residues mod p. *If $q_1=3$, $q_2>p^{1/2}/37.5$, $q_3\geqq(p/5)^{1/2}$, then, for $1\leqq A<10p^{1/2}$, we have*

$$N > 20p(A+1)/81 - 0.05314pp^{1/2}.$$

Proof. We have

$$N > (S_3 - S_9) + (S_{27} - S_{81}) - S_{3q_2} - \sum_{(p/5)^{1/2}\leqq q\leqq(p-1)/6} S_{3q},$$

where q runs over the primes, since all positive integers less than $3(p/5)^{1/2}$ which are not divisible by q_2 but are exactly divisible by either 3 or $3^3=27$ are quadratic non-residues, mod p.

Since $2A\times 27<540p^{1/2}<p$, we have, by Lemma 21,

$$\begin{aligned} S_3 - S_9 + S_{27} &> (1/3 - 1/9 + 1/27)p(A+1) - (A+1)^2 - 9A - 3 \\ &= (7/27)p(A+1) - (A+1)^2 - 9A - 3. \end{aligned} \tag{1}$$

Since $81(A+1)<810(p^{1/2}+1)<p$, we have, by Lemma 23,

$$S_{81} < p(A+1)/81 - (A+1)^2 + 5p/486. \tag{2}$$

By the same lemma, we have

$$S_{3q_2} < \frac{p^2}{9q_2^2} \times \frac{1}{4} + \frac{5p}{18q_2} = \frac{p^2}{36q_2^2} + \frac{5p}{18q_2} < 0.03909pp^{1/2}, \tag{3}$$

since $\{x\} - \{x\}^2 \leqq 1/4$ for all x. Similarly,

$$\sum_{(p/5)^{1/2} \leqq q \leqq (p-1)/6} S_{3q} < \sum_{(p/5)^{1/2} \leqq q \leqq (p-1)/6} \left(\frac{p^2}{36q^2} + \frac{5p}{18q} \right)$$
$$< \frac{p^2}{36} \frac{1.2142}{(p/5)^{1/2} \log (p/5)^{1/2}} + \frac{5p}{18} \left(\log \frac{(5p)^{1/2}}{6} + \left(\frac{5}{p} \right)^{1/2} \right),$$

by Lemma 19. Hence

$$\begin{aligned} \sum_{(p/5)^{1/2} \leqq q \leqq (p-1)/6} S_{3q} &< \frac{1.2142(5)^{1/2}}{36 \times 6.103} pp^{1/2} + \frac{5pp^{1/2}}{18000} (5.9208 + 0.0023) \\ &= \frac{2.71508}{219.708} pp^{1/2} + \frac{5 \times 5.9231}{18000} pp^{1/2} \\ &= (0.01236 + 0.00165)pp^{1/2} \\ &= 0.01401pp^{1/2}. \end{aligned} \tag{4}$$

By (1), (2), (3) and (4), we have

$$\begin{aligned} N &> 20p(A+1)/81 - 9A - 3 - 5p/486 - (0.03909 + 0.01401)pp^{1/2} \\ &> 20p(A+1)/81 - 9A - 3 - 0.00002pp^{1/2} - 0.0531)pp^{1/2} \\ &> 20p(A+1)/81 - 0.05314pp^{1/2}. \end{aligned}$$

LEMMA 25. *Let* $p \equiv 1 \pmod 4$ *and*

$$S = \sum_{a=1}^{A} \sum_{n=1, 3 \nmid n}^{a} \left(\frac{n}{p} \right).$$

Then under the conditions of Lemma 24, *we have, for* $p^{1/2} < A < 10p^{1/2}$,

$$S < 41Ap^{1/2}/81 + 0.21256p.$$

Proof. We have

$$\begin{aligned} S &= \sum_{a=1}^{A} \sum_{n=1}^{a} \left(\frac{n}{p} \right) + \sum_{a=1}^{A} \sum_{n=1}^{[a/3]} \left(\frac{n}{p} \right) \\ &\leqq \sum_{a=1}^{A} \sum_{n=1}^{a} \left(\frac{n}{p} \right) + 2 \sum_{a=1}^{[A/3-1]} \sum_{n=1}^{a} \left(\frac{n}{p} \right) + 3 \left| \sum_{n=1}^{[A/3]} \left(\frac{n}{p} \right) \right|. \end{aligned}$$

As in the proof of Lemma 5 of I, we have

$$p^{1/2}\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{n}{p}\right) = \sum_{x=1}^{(p-1)/2}\left(\frac{x}{p}\right)\left(\frac{\sin \pi(A+1)x/p}{\sin \pi x/p}\right)^2$$
$$= S_1(A)/2 - N(A).$$

Therefore, by Lemmas 21 and 24,

$$\begin{aligned}
S &\leqq \frac{1}{2p^{1/2}}S_1(A) - \frac{1}{p^{1/2}}N(A)\\
&\quad + \frac{3}{2p^{1/2}}S_1\left(\left[\frac{A}{3}-1\right]\right) - \frac{3}{p^{1/2}}N\left(\left[\frac{A}{3}-1\right]\right) + A\\
&< \frac{1}{2p^{1/2}}\left(p(A+1) - (A+1)^2 - \frac{40}{81}p(A+1) + 0.10628pp^{1/2}\right)\\
&\quad + \frac{3}{2p^{1/2}}\left(p\left[\frac{A}{3}\right] - \left[\frac{A}{3}\right]^2 - \frac{40}{81}p\left[\frac{A}{3}\right] + 0.10628pp^{1/2}\right) + A\\
&< \frac{1}{2p^{1/2}}\left(\frac{41}{81}p(A+1) - (A+1)^2 + 0.10628pp^{1/2}\right)\\
&\quad + \frac{3}{2p^{1/2}}\left(\frac{41}{81}p\frac{A}{3} + 0.10628pp^{1/2}\right)\\
&< \frac{41}{81}Ap^{1/2} + 0.21256p.
\end{aligned}$$

LEMMA 26. *Let* $p\equiv 1 \pmod 4$, *then under the conditions of Lemma* 24, *we have*

$$q_3 < 3p^{1/2}.$$

Proof. We have, for $A<q_3$,

$$\sum_{a=1}^{A}\sum_{n=1,3\nmid n}^{a}\left(\frac{n}{p}\right) > \sum_{a=1}^{A}\sum_{n=1,3\nmid n}^{a}1 - \sum_{a=1}^{A}\sum_{n=1,q_2|n}^{a}1 + \sum_{a=1}^{A}\sum_{n=1,3q_2|n}^{a}1$$
$$> A(A+1)/3 - (1/3q_2)A(A+1) - 2A.$$

By Lemma 7 of I, we have $q_3<7.5p^{1/2}<10p^{1/2}$, therefore for $A=q_3-1$ we have, by Lemma 25 (if $A<p^{1/2}$, the result is trivial),

$$\begin{aligned}
(A(A+1)/3)(1-1/q_2) - 2A &< 41Ap^{1/2}/81 + 0.21256p,\\
(A(A+1)/3)(1-0.0375) - 2A &< 0.5062Ap^{1/2} + 0.21256p,\\
0.3208A^2 - 1.6792A &< 0.5062Ap^{1/2} + 0.21256p,\\
0.3208A^2 - 0.5079Ap^{1/2} - 0.21256p &< 0.
\end{aligned}$$

Hence

$$A < (1/0.6416)(0.5079 + (0.5079^2 + 4 \times 0.3208 \times 0.21256)^{1/2})p^{1/2},$$

which implies

$$q_3 < 3p^{1/2}.$$

5. Lemmas concerning sums.

LEMMA 27. *We have, for $q>0$,*

$$\frac{(A+2)(A-q)}{2q} < \sum_{a=1}^{A}\left[\frac{a}{q}\right] < \frac{1}{2q}\left(A - \frac{(q-2)}{2}\right)^2.$$

Proof. We have

$$\begin{aligned}\sum_{a=1}^{A}\left[\frac{a}{q}\right] &= \sum_{q\leqq a<2q} + \sum_{2q\leqq a<3q} + \cdots + \sum_{([A/q]-1)q\leqq a<[A/q]q}\left[\frac{A}{q}-1\right] \\ &\quad + \sum_{[A/q]q\leqq a\leqq A}\left[\frac{A}{q}\right] \\ &= q\left(1+2+\cdots+\left[\frac{A}{q}-1\right]\right) + \left(A - \left[\frac{A}{q}\right]q + 1\right)\left[\frac{A}{q}\right] \\ &= \frac{1}{2}\, q\left[\frac{A}{q}\right]\left(\left[\frac{A}{q}\right]-1\right) + \left(A - \left[\frac{A}{q}\right]q+1\right)\left[\frac{A}{q}\right] \\ &= \frac{1}{2}\left[\frac{A}{q}\right]\left(2A+2-\left[\frac{A}{q}\right]q - q\right) \\ &= \frac{1}{2q}(A-\vartheta q)(A-q+\vartheta q+2) \\ &= \frac{1}{2q}\left(\left(A-\frac{q}{2}+1\right)^2 - \left(\vartheta q + 1 - \frac{q}{2}\right)^2\right), \qquad \vartheta = \frac{A}{q} - \left[\frac{A}{q}\right],\end{aligned}$$

which lies between $(A+2)(A-q)/2q$ and $(A-q/2+1)^2/2q$.

LEMMA 28 (EULER'S SUMMATION FORMULA). *Let $b_1(x) = x - [x] - 1/2$ and*

$$b_2(x) = \frac{1}{12} + \int_0^x b_1(x)dx.$$

Let $b>a$ and let $g(x)$ with its first two derivatives be continuous for $a\leqq x<b$. Then

$$\begin{aligned}\sum_{a\leqq m<b} g(m) &= \int_a^b g(x)dx + g(b)b_1(-b) - g(a)b_1(-a) \\ &\quad + g'(b)b_2(-b) - g'(a)b_2(-a) - \int_a^b g''(x)b_2(-x)dx.\end{aligned}$$

LEMMA 29. *For* $a-\lambda q>3q\geqq 9$,

$$\sum_{\nu=1}^{[(a-\lambda q)/3q]} li\,\frac{a}{3\nu+\lambda} < \frac{a}{3}\log\frac{\log(a/(3+\lambda))}{\log q} + \frac{(3\lambda+10.7322)a}{12(3+\lambda)^3\log(a/(3+\lambda))}$$

$$- \frac{3q^2b_2(-(a-\lambda q)/3q)}{a\log q} + \frac{a}{3q}\,li\,q$$

$$- \left(\frac{1}{2}+\frac{\lambda}{3}\right)li\,\frac{a}{3+\lambda} + li\,qb_1\left(-\frac{a-\lambda q}{3q}\right) + R,$$

where $R=li\,q$ *or* 0 *according as* $3q$ *does or does not divide* $a-\lambda q$.

Proof. Let $g(x)=li\,a/(3x+\lambda)$. Then

$$g'(x) = -\frac{3a}{(3x+\lambda)^2\log(a/(3x+\lambda))},$$

$$g''(x) = \frac{9a(2\log(a/(3x+\lambda))-1)}{(3x+\lambda)^3\log^2(a/(3x+\lambda))},$$

and for $1\leqq 3x+\lambda\leqq(1/e)a$, $g''(x)$ is monotonically decreasing, since the function $(2u-1)/u^2$ is monotonically decreasing for $u>1$.

By Lemma 28,

$$\sum_{\nu=1}^{[(a-\lambda q)/3q]} li\,\frac{a}{3\nu+\lambda} = \int_1^{(a-\lambda q)/3q} li\,\frac{a}{3t+\lambda}\,dt + \frac{1}{2}\,li\,\frac{a}{3+\lambda}$$

$$+ li\,qb_1\left(-\frac{a-\lambda q}{3q}\right) + \frac{a}{4(3+\lambda)^2\log(a/(3+\lambda))}$$

$$- \frac{3ab_2(-(a-\lambda q)/3q)}{(a/q)^2\log q}$$

$$- \int_1^{(a-\lambda q)/3q}\frac{9a(2\log(a/(3x+\lambda))-1)b_2(-x)}{(3x+\lambda)^3\log^2(a/(3x+\lambda))}dx + R$$

$$= \frac{a}{3}\log\frac{\log(a/(3+\lambda))}{\log q} + \frac{a}{3q}li\,q - \frac{1}{3}(3+\lambda)\,li\,\frac{a}{3+\lambda}$$

$$+ \frac{1}{2}\,li\,\frac{a}{3+\lambda} + li\,qb_1\left(-\frac{a-\lambda q}{3q}\right)$$

$$+ \frac{a}{4(3+\lambda)^2\log(3/(3+\lambda))} - \frac{3q^2b_2(-(a-\lambda q)/3q)}{a\log q}$$

$$+ \frac{3^{1/2}a\theta}{12(3+\lambda)^3\log(a/(3+\lambda))} + R,$$

where $\theta\leqq 1$, since

$$\int_1^{(a-\lambda q)/3q} li\frac{a}{3t+\lambda}\,dt$$
$$= \frac{1}{3}\left((3t+\lambda)\,li\frac{a}{3t+\lambda}\right)_1^{(a-\lambda q)/3q}$$
$$-\frac{a}{3}\int_1^{(a-\lambda q)/3q}\frac{d(a/(3t+\lambda))/dt}{a/(3t+\lambda)\log(a/(3t+\lambda))}\,dt$$
$$= \frac{1}{3}\left((3t+\lambda)\,li\frac{a}{3t+\lambda}\right)_1^{(a-\lambda q)/3q}-\frac{a}{3}\left(\log\log\frac{a}{3t+\lambda}\right)_1^{(a-\lambda q)/3q}$$
$$= \frac{a}{3q}li\,q-\frac{1}{3}(3+\lambda)\,li\frac{a}{3+\lambda}+\frac{a}{3}\log\frac{\log(a/(3+\lambda))}{\log q}$$

and

$$\left|\int_0^x b_2(-x)dx\right| \leqq \frac{3^{1/2}}{216}\ (4).$$

The lemma follows immediately.

LEMMA 30. *Under the conditions of Lemma* 29, *and the condition* $\lambda>0$, *we have*

$$\sum_{a=\lambda q}^{A}\sum_{\nu=0}^{[(a-\lambda q)/3q]} li\frac{a}{3\nu+\lambda} < \frac{1}{6}A(A+2)\log\log\frac{A}{3+\lambda}$$
$$-\frac{1}{6}(A^2+A+(3+\lambda)q)\log\log q+(A+1)\,li\frac{A}{\lambda}-\frac{(2\lambda+3)A}{6}\,li\frac{A}{3+\lambda}$$
$$+\frac{li\,q}{24q}(4A^2+8A+(2\lambda-3)^2q^2+4)+\frac{3\lambda+10.7322}{12(\lambda+3)^3}\,\frac{A}{\log A/(3+\lambda)}$$
$$+\frac{q^2}{8\log q}\left(\log\frac{A}{(3+\lambda)q}+\frac{1}{(3+\lambda)q}\right)-\lambda\int_q^{A/\lambda}\frac{xdx}{\log x}$$
$$+\frac{2\lambda(3+\lambda)^2+3\lambda+10.7322}{12(3+\lambda)}\int_q^{A/(3+\lambda)}\frac{xdx}{\log x}.$$

Proof. By Lemma 29, we have

$$\sum_{a=\lambda q}^{A}\sum_{\nu=0}^{[(a-\lambda q)/3q]} li\frac{a}{3\nu+\lambda} = \sum_{a=\lambda q}^{A} li\frac{a}{\lambda}+\sum_{a=(3+\lambda)q}^{A}\sum_{\nu=1}^{[(a-\lambda q)/3q]} li\frac{a}{3\nu+\lambda}$$

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(4) For $0<x\leqq 1$, $\int_0^x b_2(-x)dx=\int_0^x((x^2-x)/2+1/12)dx=(x/12)(2x^2-3x+1)$, which takes its extrema at $x=(3\pm 3^{1/2})/6$. For $x=1$, the integral equals zero. Since $b_2(x-1)=b_2(x)$, we have the required formula.

$$
\begin{aligned}
&< \int_{\lambda q}^{A} li\frac{x}{\lambda}dx + li\frac{A}{\lambda} + \sum_{a=(3+\lambda)q}^{A} \frac{a}{3}\log\frac{\log(a/(3+\lambda))}{\log q} \\
&+ \frac{3\lambda+10.7322}{12(3+\lambda)^3}\left(\int_{(3+\lambda)q}^{A}\frac{xdx}{\log(x/(3+\lambda))} + \frac{A}{\log(A/(3+\lambda))}\right) \\
&- \sum_{a=(3+\lambda)q}^{A}\frac{3q^2}{a\log q}b_2\left(-\frac{a-\lambda q}{3q}\right) \\
&+ \frac{1}{6q}(A+\lambda q+3q)(A-\lambda q-3q+1)\,li\,q \\
&- \left(\frac{1}{2}+\frac{\lambda}{3}\right)\int_{(3+\lambda)q}^{A} li\frac{x}{3+\lambda}dx \\
&+ li\,q\sum_{a=(3+\lambda)q}^{A}\left(b_1\left(-\frac{a-\lambda q}{3q}\right)+R'\right)
\end{aligned}
$$

where $R'=1$ or 0 according as $3q$ does or does not divide $a-\lambda q$.

It is plain that

$$
\int_{\lambda q}^{A} li\frac{x}{\lambda}dx = A\,li\frac{A}{\lambda} - \lambda q\,li\,q - \int_{\lambda/q}^{A}\frac{xdx}{\lambda\log x/\lambda},
$$

$$
\begin{aligned}
\sum_{a=(3+\lambda)q}^{A}\frac{a}{3}\log\log\frac{a}{3+\lambda} &< \frac{1}{3}\int_{(3+\lambda)q}^{A} x\log\log\frac{x}{3+\lambda}dx + \frac{A}{3}\log\log\frac{A}{3+\lambda} \\
&= \frac{1}{6}\left(A^2\log\log\frac{A}{3+\lambda} - (3+\lambda)^2q^2\log\log q\right) \\
&\quad + \frac{A}{3}\log\log\frac{A}{3+\lambda} - \frac{1}{6}\int_{(3+\lambda)q}^{A}\frac{xdx}{\log x/(3+\lambda)} \\
&\quad - \sum_{a=(3+\lambda)q}^{A}\frac{b_2(-(a-\lambda q)/3q)}{a} < \frac{1}{24}\sum_{a=(3+\lambda)q}^{A}\frac{1}{a} \\
&< \frac{1}{24}\left(\log\frac{A}{(3+\lambda)q} + \frac{1}{(3+\lambda)q}\right),
\end{aligned}
$$

since $b_2(x) \geqq -1/24$,

$$
\int_{(3+\lambda)q}^{A} li\frac{x}{3+\lambda}dx = A\,li\frac{A}{3+\lambda} - (3+\lambda)q\,li\,q - \int_{(3+\lambda)q}^{A}\frac{xdx}{(3+\lambda)\log x/(3+\lambda)}
$$

and

$$
\sum_{a=(3+\lambda)q}^{A}\left(b_1\left(-\frac{a-\lambda q}{3q}\right)+R'\right) = \sum_{n=0}^{A-(3+\lambda)q}\left(b_1\left(-\frac{n}{3q}\right)+R''\right) = S,
$$

say, where $R''=1$ or 0 according as $3q|n$ or $3q\nmid n$. Clearly,

$$S = \sum_{n=0}^{A-(3+\lambda)q}\left(-\frac{n}{3q}-\left[-\frac{n}{3q}\right]-\frac{1}{2}+R''\right)$$
$$= \sum_{n=0}^{A-(3+\lambda)q}\left(-\frac{n}{3q}+\left[\frac{n}{3q}\right]+\frac{1}{2}\right),$$

since $[-n/3q]=-[n/3q]+R''-1$. By Lemma 27,

$$\begin{aligned} S &< -\frac{1}{6q}(A-(3+\lambda)q)(A-(3+\lambda)q+1) \\ &\quad + \frac{1}{6q}\left(A-(3+\lambda)q-\frac{3q-2}{2}\right)^2+\frac{1}{2}(A-(3+\lambda)q+1) \\ &= \frac{1}{6q}(A-\lambda q)+\frac{(3q-2)^2}{24q}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{a=\lambda q}^{A}\sum_{\nu=0}^{[(a-\lambda q)/3q]} li\,\frac{a}{3\nu+\lambda} &< A\,li\,\frac{A}{\lambda}-\lambda q\,li\,q-\int_{\lambda q}^{A}\frac{xdx}{\lambda\log x/\lambda}+li\,\frac{A}{\lambda} \\ &\quad -\frac{1}{6}(A+\lambda q+3q)(A-\lambda q-3q+1)\log\log q \\ &\quad +\frac{1}{6}\left(A^2\log\log\frac{A}{3+\lambda}-(3+\lambda)^2q^2\log\log q\right)+\frac{A}{3}\log\log\frac{A}{3+\lambda} \\ &\quad -\frac{1}{6}\int_{(3+\lambda)q}^{A}\frac{xdx}{\log x/(3+\lambda)} \\ &\quad +\frac{3\lambda+10.7322}{12(3+\lambda)^3}\left(\int_{(3+\lambda)q}^{A}\frac{xdx}{\log x/(3+\lambda)}+\frac{A}{\log A/(3+\lambda)}\right) \\ &\quad +\frac{q^2}{8\log q}\left(\log\frac{A}{(3+\lambda)q}+\frac{1}{(3+\lambda)q}\right) \\ &\quad +\frac{1}{6q}(A+\lambda q+3q)(A-\lambda q-3q+1)\,li\,q \\ &\quad -\left(\frac{1}{2}+\frac{\lambda}{3}\right)\left(A\,li\,\frac{A}{3+\lambda}\right)-(3+\lambda)q\,li\,q \\ &\quad -\int_{(3+\lambda)q}^{A}\frac{xdx}{(3+\lambda)\log x/(3+\lambda)}+li\,q\left(\frac{A-\lambda q}{6q}+\frac{(3q-2)^2}{24q}\right) \end{aligned}$$

$$= \frac{1}{6} A(A+2) \log\log \frac{A}{3+\lambda} - \frac{1}{6}(A^2 + A + (3+\lambda)q) \log\log q + (A+1)\, li \frac{A}{\lambda}$$
$$- \frac{2\lambda+3}{6} A\, li \frac{A}{3+\lambda} + \frac{li\, q}{24q}(4A^2 + 8A + (2\lambda-3)^2 q^2 + 4)$$
$$+ \frac{3\lambda + 10.7322}{12(3+\lambda)^3} \frac{A}{\log (A/(3+\lambda))} + \frac{q^2}{8 \log q}\left(\log \frac{A}{(3+\lambda)q} + \frac{1}{(3+\lambda)q}\right)$$
$$- \int_{\lambda q}^{A} \frac{x dx}{\lambda \log x/\lambda} + \frac{2\lambda(3+\lambda)^2 + 3\lambda + 10.7322}{12(3+\lambda)} \int_{q}^{A/(3+\lambda)} \frac{x dx}{\log x}.$$

LEMMA 31. *Under the conditions of Lemma* 29.

$$\sum_{a=q}^{A}\left(\sum_{\nu=0}^{[(a-q)/3q]} li \frac{a}{3\nu+1} + \sum_{\nu=0}^{[(a-2q)/3q]} li \frac{a}{3\nu+2}\right)$$
$$< \frac{1}{6} A(A+2) \log\left(\log \frac{A}{4} \log \frac{A}{5}\right) - \frac{1}{6}(2A^2 + 2A + 9q) \log\log q$$
$$+ \frac{li\, q}{12q}(4A^2 + 8A + q^2 + 4) + \frac{0.03A}{\log A/5}$$
$$+ \frac{q^2}{8 \log q}\left(\log \frac{A^2}{20q^2} + \frac{q}{20q}\right) + A^2\left(\int_{1/2}^{1} \frac{(1-x)dx}{\log Ax} + \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax}\right.$$
$$\left. + \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472x)dx}{\log Ax} + 0.1016 \int_{q/A}^{1/5} \frac{x dx}{\log Ax}\right) + li\, A + li \frac{A}{2}.$$

Proof. By Lemma 30, the sum in question is less than

$$\frac{1}{6} A(A+2) \log\left(\log \frac{A}{4} \log \frac{A}{5}\right) - \frac{1}{6}(2A^2 + 2A + 9q) \log\log q$$
$$+ (A+1)\, li\, A + (A+1)\, li \frac{A}{2} - \frac{5}{6} A\, li \frac{A}{4} - \frac{7}{6} A\, li \frac{A}{5}$$
$$+ \frac{li\, q}{12q}(4A^2 + 8A + q^2 + 4) + \frac{0.03A}{\log (A/5)}$$
$$+ \frac{q^2}{8 \log q}\left(\log \frac{A^2}{20q^2} + \frac{9}{20q}\right) - \int_{q}^{A} \frac{x dx}{\log x} - \int_{2q}^{A} \frac{x dx}{2 \log x/2}$$
$$+ \frac{45.7322}{48} \int_{q}^{A/4} \frac{x dx}{\log x} + \frac{116.7322}{60} \int_{q}^{A/5} \frac{x dx}{\log x}.$$

Since

$$A\,li\,A + A\,li\frac{A}{2} - \frac{5}{6}A\,li\frac{A}{4} - \frac{7}{6}A\,li\frac{A}{5}$$

$$= A\int_{A/2}^{A}\frac{dx}{\log x} + 2A\int_{A/4}^{A/2}\frac{dx}{\log x} + \frac{7}{6}A\int_{A/5}^{A/4}\frac{dx}{\log x}$$

and

$$-\int_{q}^{A}\frac{xdx}{\log x} - \int_{2q}^{A}\frac{xdx}{2\log x/2} + \frac{45.7322}{48}\int_{q}^{A/4}\frac{xdx}{\log x} + \frac{116.7322}{60}\int_{q}^{A/5}\frac{xdx}{\log x}$$

$$= -\int_{A/2}^{A}\frac{xdx}{\log x} - 3\int_{A/4}^{A/2}\frac{xdx}{\log x} - 2.0472\int_{A/5}^{A/4}\frac{xdx}{\log x}$$

$$- 0.1016\int_{q}^{A/5}\frac{xdx}{\log x},$$

we have the lemma by replacing x in the integrals by Ax.

6. The growth of smaller quadratic non-residues.

LEMMA 32. *Let* $p\equiv 1 \pmod 4$, $p>10^6$, $q_1=3$. *Then* $5q_2q_3<p$.

Proof. Suppose on the contrary that $5q_2q_3\geqq p$. Then

$$q_3 > (p/5)^{1/2}. \tag{1}$$

By Lemma 7 of I, we have $q_3\leqq 4p^{1/2}/(1-1/3)(1-1/5)=7.5p^{1/2}$.

If $q_2<p^{1/2}/37.5$, the lemma is immediate. If $q_2\geqq p^{1/2}/37.5$, by Lemma 26,

$$q_3 < 3p^{1/2}. \tag{2}$$

The lemma follows for $q_2\leqq p^{1/2}/15$. Then we may assume that

$$q_2 > p^{1/2}/15. \tag{3}$$

By Lemma 10 of I, we have

$$\frac{1}{2}\sum_{a=1}^{A}\sum_{n=1,3\nmid n}^{a}\left(1-\left(\frac{n}{p}\right)\right) \leqq \sum_{a=q_3}^{A}\left(\sum_{q_3\leqq q\leqq a}\left[\frac{a}{q}\right] - \sum_{q_3\leqq q\leqq a/3}\left[\frac{a}{3q}\right]\right)$$

$$+ \sum_{a=q_2}^{A}\left(\left[\frac{a}{q_2}\right]-\left[\frac{a}{3q_2}\right]\right)$$

$$= \sum_{a=q_3}^{A}\left(\sum_{\nu=1}^{[a/q_3]}\pi\left(\frac{a}{\nu}\right) - \left[\frac{a}{q_3}\right](\pi(q_3)-1)\right.$$

$$\left. - \sum_{\nu=1}^{[a/3q_3]}\pi\left(\frac{a}{3\nu}\right) + \left[\frac{a}{3q_3}\right](\pi(q_3)-1)\right)$$

$$+ \sum_{a=q_2}^{A}\left(\left[\frac{a}{q_2}\right]-\left[\frac{a}{3q_2}\right]\right).$$

Denoting the first side of the inequality by S_0, we have

$$
\begin{aligned}
(4)\qquad S_0 &< \sum_{a=q_2}^{A} \sum_{\nu=1,3\nmid\nu}^{[a/q_3]} \pi\left(\frac{a}{\nu}\right) - \sum_{a=q_3}^{A}\left(\left[\frac{a}{q_3}\right]-\left[\frac{a}{3q_3}\right]\right)(\pi(q_3)-1) \\
&\quad + \sum_{a=q_2}^{A}\left(\left[\frac{a}{q_2}\right]-\left[\frac{a}{3q_2}\right]\right) = S' - S'' + S''', \text{ say.}
\end{aligned}
$$

(1) Suppose that $p > 10^8$. Let us take $A = [10p^{1/2}]$. By Lemma 25,

$$
\begin{aligned}
(5)\qquad S_0 &> (A^2/3 - 41Ap^{1/2}/81 - 0.21256p)/2 \\
&> (33.32 - 5.1 - 0.22)p/2 = 14p
\end{aligned}
$$

since

$$A^2/3 > (100p - 20p^{1/2})/3 = 33.32p + 4p/300 - 20p^{1/2}/3 > 33.32p.$$

By (3),

$$
\begin{aligned}
(6)\qquad S''' &< \sum_{a=q_2}^{A}\left(\frac{a}{q_2} - \frac{a}{3q_2} + 1\right) < \frac{A(A+1)}{3q_2} + A \\
&< 50(10p^{1/2}+1) + 10p^{1/2} = 510p^{1/2} + 50 \\
&< 0.06p.
\end{aligned}
$$

By Lemma 2 of I, we have, since $q_3 > (p/5)^{1/2} \geqq 10^4/5^{1/2} > 51^2$,

$$S'' > (0.9607\, li\, q_3 - 2.7)\sum_{a=q_3}^{A}\left(\left[\frac{a}{q_3}\right]-\left[\frac{a}{3q_3}\right]\right).$$

By Lemma 27 and (2),

$$
\begin{aligned}
&\sum_{a=q_3}^{A}\left(\left[\frac{a}{q_3}\right]-\left[\frac{a}{3q_3}\right]\right) \\
&\qquad > \frac{1}{2q_3}\left((A+2)(A-q_3) - \frac{1}{3}\left(A+1-\frac{3q_3}{2}\right)^2\right) \\
&\qquad = \frac{1}{2q_3}\left(\frac{2}{3}A^2 + \frac{4}{3}A - \frac{3}{4}q_3^2 - q_3 - \frac{1}{3}\right) \\
&\qquad > \frac{p}{q_3}\left(\frac{100}{3} - \frac{1}{3\times 10^8} - \frac{3^3}{8} - \frac{3}{2\times 10^4} - \frac{1}{6\times 10^8}\right) \\
&\qquad > \frac{29.957}{q_3}p.
\end{aligned}
$$

Then

$$(7)\qquad S'' > (29.957/q_3)(0.9607\, li\, q_3 - 2.7) > (28.779\, li\, q_3/q_3 - 0.03)p.$$

Finally, by Lemma 2 of I and Lemma 31,

$$S' < 1.0376 \sum_{a=q_3}^{A} \sum_{\nu=1,3\nmid\nu}^{a/q_3} (li\, a/\nu + 1.85)$$

$$< 1.0376\left\{\frac{1}{6} A(A+2) \log\left(\log\frac{A}{4} \log\frac{A}{5}\right)\right.$$

$$- \frac{1}{6}(2A^2 + 2A + 9q_3) \log\log q_3 + \frac{li\, q_3}{12q_3}(4A^2 + 8A + q_3^2 + 4)$$

$$+ \frac{0.03A}{\log A/5} + \frac{q_3^2}{8\log q_3}\left(\log\frac{A^2}{20q_3^2} + \frac{9}{20q_3}\right) + A^2\left(\int_{1/2}^{1} \frac{(1-x)dx}{\log Ax}\right.$$

$$+ \int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} + \int_{1/5}^{1/4} \frac{(1.1667 - 2.0472)dx}{\log Ax}$$

$$\left.\left. + 0.1016 \int_{q_3/A}^{1/5} \frac{xdx}{\log Ax}\right) + li\, A + li\frac{A}{2}\right\}$$

$$+ 1.91956 \sum_{a=q_3}^{A} \left(\left[\frac{a}{q_3}\right] - \left[\frac{a}{3q_3}\right]\right).$$

Since

$$\frac{1}{6} A(A+2)\log\left(\log\frac{A}{4}\log\frac{A}{5}\right) - \frac{1}{6}(2A^2 + 2A + 9q_3)\log\log q_3$$

$$< \frac{1.0001}{6} A^2\left(\log\frac{\log(5/2)p^{1/2}\log 2p^{1/2}}{(\log p^{1/2} - (\log 5)/2)^2} + \frac{2\log\log(p/5)^{1/2}}{[10p^{1/2}]}\right)$$

$$< \frac{1.0001}{6} A^2\left(\log\frac{(4\log 10 + \log 2.5)(4\log 10 + \log 2)}{(4\log 10 - (\log 5)/2)^2} + 0.0002\right)$$

$$< \frac{1.0001}{6} A^2\left(\log\frac{10.267 \times 9.9035}{8.40562^2} + 0.0002\right)$$

$$= \frac{1.0001}{6} A^2(\log 1.42 + 0.0002)$$

$$< \frac{0.351}{6} A^2 < 5.85p.$$

$$\frac{li\, q_3}{12q_3}(4A^2 + 8A + q_3^2 + 4) < \frac{\log q_3}{q_3} p\left(\frac{100}{3} + \frac{2\times 10}{3\times 10^4} + \frac{3^2}{12} + \frac{1}{3\times 10^8}\right)$$

$$< 34.0841\, li\, q_3 p/q_3,$$

$$\frac{0.03A}{\log A/5} < \frac{0.3p^{1/2}}{\log(2p^{1/2} - 1)} = \frac{0.3p}{10^4\log(2\times 10^4 - 1)} < 0.00001p,$$

$$\frac{q_3^2}{8 \log q_3}\left(\log \frac{A^2}{20q_3^2} + \frac{9}{20q_3}\right) < \frac{9p}{8 \log (3p^{1/2})}\left(\log \frac{10^2 \times 5}{20} + \frac{9(5)^{1/2}}{20 \times 10^4}\right)$$
$$< \frac{9}{8 \times 10.30895}(3.21889 + 0.0002)p$$
$$< 0.36p$$

(since $(p/5)^{1/2} < q_3 < 3p^{1/2}$).

$$\int_{1/2}^{1} \frac{(1-x)dx}{\log Ax} < \frac{1}{\log A/2}\int_{1/2}^{1}(1-x)dx = \frac{1}{8 \times 10.81977} < 0.0116,$$

$$\int_{1/4}^{1/2} \frac{(2-3x)dx}{\log Ax} < \frac{7}{32 \log A/4} < \frac{7}{32 \times 10.126635} < 0.0217,$$

$$\int_{1/5}^{1/4} \frac{(1.1667 - 2.0472x)dx}{\log Ax} < \frac{1}{\log A/5}\left(\frac{1.1667}{20} - \frac{9 \times 2.0472}{800}\right) < 0.00357,$$

$$0.1016\int_{q_3/A}^{1/5} \frac{xdx}{\log Ax} < \frac{0.00204}{\log q_3} < 0.0001,$$

$$li\, A + li\, A/2 < 2\, li\, A < 2 \times \frac{10p^{1/2}}{\log 10p^{1/2} - 2}\,(^5) < \frac{2p}{10^3(\log 10^5 - 2)}$$
$$< 0.0003p$$

and

$$\sum_{a=q_3}^{A}\left(\left[\frac{a}{q_3}\right] - \left[\frac{a}{3q_3}\right]\right) \leqq \frac{A(A+1)}{3q_3} + A < 0.0085p.$$

We have

$$\begin{aligned} S' < {} & 1.0376p\{5.85 + 34.0841\, li\, q_3/q_3 + 0.00001p + 0.36 \\ & + 10^2(0.0116 + 0.0217 + 0.0036 + 0.0001) + 0.0003\} \\ & + 1.91956 \times 0.0085p \\ < {} & p(1.0376 \times 10 + 0.02 + 1.0376 \times 34.084)\, li\, q_3/q_3) \\ < {} & p(10.40 + 35.366\, li\, q_3/q_3). \end{aligned} \tag{8}$$

By (4), (5), (6), (7), and (8), we have $14p < p(10.40 + 35.366\ li\ q_3/q_3) - (28.779\ li\ q_3/q_3 - 0.03)p + 0.06p$, that is,

$$14 < 10.40 + 0.03 + 0.06 + 6.59\, li\, q_3/q_3$$
$$< 10.49 + 6.59 \times 1/(\log(10^4/5^{1/2}) - 2) < 12,$$

which is absurd.

(5) For $e^4 \leqq x$, $li\ x < x/(\log x - 2)$. In fact, the inequality is true for $x = e^4$, and $d\ li\ x/dx \leqq d(x/(\log x - 2))/dx$.

(2) Suppose $10^6<p<10^8$. We take $A=[6p^{1/2}]$. By Lemma 25,

$$(5') \quad \begin{aligned} S_0 &> (A^2/3 - 41Ap^{1/2}/81 - 0.21256p)/2 \\ &> (12 - 0.004 - 3.0371 - 0.21256)p/2 > 4.3731p. \end{aligned}$$

By Lemma 27

$$\begin{aligned} S''' &< \frac{1}{2q_2}\left(\left(A - \frac{q_2}{2} + 1\right)^2 - \frac{(A+2)(A-3q_2)}{3}\right) \\ &= \frac{1}{2q_2}\left(\frac{2}{3}A^2 + \frac{4}{3}A + \frac{(q_2+2)^2}{4}\right) \\ &= \frac{1}{q_2}\left(\frac{1}{3}A^2 + \frac{2}{3}A + \frac{(q_2+2)^2}{8}\right) < \frac{p}{q_2}\left(12 + \frac{4}{10^3} + \frac{(q_2+2)^2}{8p}\right) \\ &< 5q_3\left(12.004 + \frac{(p+10q_3)^2}{200pq_3^2}\right), \end{aligned}$$

for $q_2>p/5q_3$, and the function is decreasing for $q_2<q_3<3p^{1/2}$ (by (2)). Since

$$\frac{(p+10q_3)^2}{200pq_3^2} = \frac{1}{200p}\left(\frac{p}{q_3}+10\right)^2 < \frac{1}{200}\left(5^{1/2}+\frac{10}{p^{1/2}}\right)^2 < \frac{1}{200}\left(5^{1/2}+\frac{1}{100}\right)^2 < 0.0253,$$

$$(6') \qquad S''' < 60.1465q_3.$$

Since $11<(p/5)^{1/2}<q_3<3p^{1/2}<10^6$, we have, by Lemma 18, $S''>(li\ q_3 - li\ q_3^{1/2}-1)\sum_{a=q_3}^{A}([a/q_3]-[a/3q_3])$.

By Lemma 27 and (2), we have as in (1)

$$\begin{aligned} \sum_{a=q_3}^{A}\left(\left[\frac{a}{q_3}\right]-\left[\frac{a}{3q_3}\right]\right) &> \frac{1}{2q_3}\left(\frac{2}{3}A^2 + \frac{4}{3}A - \frac{3}{4}q_3^2 - q_3 - \frac{1}{3}\right) \\ &> \frac{p}{2q_3}\left(\frac{2}{3}\cdot 6^2 - \frac{8}{p^{1/2}} + \frac{4}{3}\frac{6}{p^{1/2}} - \frac{3}{4}\frac{q_3^2}{p} - \frac{q_3}{p^{1/2}} - \frac{1}{p}\right) \\ &> \frac{p}{2q_3}\left(24 - \frac{3}{4}\frac{q_3^2}{p} - \frac{3}{p^{1/2}} - \frac{1}{p}\right) \\ &> \frac{p}{2q_3}\left(23.9969 - \frac{3}{4}\frac{q_3^2}{p}\right) > \frac{p}{q_3}\left(11.9984 - \frac{3}{8}\frac{q_3^2}{p}\right), \end{aligned}$$

$$\frac{1}{q_3} < \left(\frac{5}{p}\right)^{1/2} < \frac{p^{1/2}}{10^3} = 0.002237,$$

$$\frac{li\ q_3^{1/2}}{q_3} < \frac{li\ (10^3/5^{1/2})^{1/2}}{10^3/5^{1/2}} < \frac{li\ 21.176}{447.21} < \frac{10.75}{447.21} < 0.02405.$$

Therefore

$$S'' > p(li\, q_3/q_3 - 0.0263)(11.9984 - 3q_3^2/8p). \tag{7'}$$

Finally, by Lemmas 17 and 31, we have

$$S' = \sum_{a=q_3}^{A} \sum_{\nu=1,3\nmid\nu}^{a/q_3} \pi\left(\frac{a}{\nu}\right) < \sum_{a=q_3}^{A} \sum_{\nu=1,3\nmid\nu}^{a/q_3} li\,\frac{a}{\nu}$$

$$< \frac{1}{6}A(A+2)\log\left(\log\frac{A}{4}\log\frac{A}{5}\right) - \frac{1}{6}(2A^2+2A+9q_3)\log\log q_3$$

$$+ \frac{li\,q_3}{12q_3}(4A^2+8A+q_3^2+4) + \frac{0.03A}{\log A/5} + \frac{q_3^2}{8\log q_3}\left(\log\frac{A^2}{20q_3^2}+\frac{9}{20q_3}\right)$$

$$+ A^2\left(\int_{1/2}^{1}\frac{(1-x)dx}{\log Ax} + \int_{1/4}^{1/2}\frac{(2-3x)dx}{\log Ax}\right.$$

$$\left. + \int_{1/5}^{1/4}\frac{(1.1667-2.0472x)dx}{\log Ax} + 0.1016\int_{q_3/A}^{1/5}\frac{xdx}{\log Ax}\right) + li\,A + li\,\frac{A}{2}$$

$$< \frac{A^2}{3}\left\{\frac{1}{2}\left(1+\frac{2}{A}\right)\log\left(\log\frac{A}{4}\log\frac{A}{5}\right) - \log\log q_3 + \frac{6\,li\,A}{A^2}\right.$$

$$+ \frac{li\,q_3}{q_3}\left(1+\frac{2}{A}+\frac{q_3^2}{4A^2}+\frac{1}{A^2}\right) + \frac{0.09}{A\log A/5}$$

$$+ \frac{3q_3^2}{8A^2\log q_3}\left(\log\frac{A^2}{20q_3^2}+\frac{9}{20q_3}\right) + 3\left(\int_{1/2}^{1}\frac{(1-x)dx}{\log Ax}\right.$$

$$+ \int_{1/4}^{1/2}\frac{(2-3x)dx}{\log Ax} + \int_{1/5}^{1/4}\frac{(1.1667-2.0472x)dx}{\log Ax}$$

$$\left.\left. + 0.1016\int_{q_3/A}^{1/5}\frac{xdx}{\log Ax}\right)\right\}.$$

Since

$$\frac{1}{A}\log\left(\log\frac{A}{4}\log\frac{A}{5}\right) < \frac{1}{5999}\log(\log 1500\log 1200)$$

$$< \frac{1}{5999}\log(7.31323\times 7.09008) < 0.0007,$$

$$\frac{6\,li\,A}{A^2} < \frac{6}{5999(\log 5999 - 2)} < 0.00016,$$

$$\frac{0.09}{A\log A/5} < \frac{0.09}{5999} < 0.00002,$$

$$\frac{3p}{8A^2\log q_3}\left(\log\frac{A^2}{20q_3^2}+\frac{9}{20q_3}\right)<\frac{3}{286\ \log 10^3/5^{1/2}}\left(\log\frac{36}{4}+\frac{9(5)^{1/2}}{20\times 10^3}\right)<0.0038,$$

$$\int_{1/2}^{1}\frac{(1-x)dx}{\log Ax}<\frac{1}{\log A/2}\int_{1/2}^{1}(1-x)dx<\frac{1}{8.006}\times\frac{1}{8}<0.0157,$$

$$\int_{1/4}^{1/2}\frac{(2-3x)dx}{\log Ax}<\frac{1}{\log A/4}\int_{1/4}^{1/2}(2-3x)dx<0.03,$$

$$\int_{1/5}^{1/4}\frac{(1.1667-2.0472x)dx}{\log Ax}<\frac{1}{\log A/5}\int_{1/5}^{1/4}(1.1667-2.0472x)dx<0.005,$$

$$0.1016\int_{q_3/A}^{1/5}\frac{xdx}{\log Ax}<0.1016\times\frac{0.002}{\log q_3}<\frac{0.1016\times 0.002}{6.103}<0.0001,$$

and the total sum of the last four integrals is less than 0.0508, we have

$$\begin{aligned} S' < 12p\Biggl\{&\frac{1}{2}\log\left(\log\frac{A}{4}\log\frac{A}{5}\right)-\log\log q_3 \\ &+\frac{li\,q_3}{q_3}\left(1+\frac{2}{5.999p^{1/2}}+\frac{q_3^2}{143.952p}+0.00001\right) \\ &+0.0038\,\frac{q_3^2}{p}+0.15328\Biggr\} \end{aligned} \tag{8'}$$

(for $0.0007+0.00016+0.00002+3\times 0.0508=0.15328$).

By (4), (5′), (6′), (7′) and (8′), we have

$$\begin{aligned} 4.3731 < 12\Biggl\{&\frac{1}{2}\log\left(\log\frac{A}{4}\log\frac{A}{5}\right)-\log\log q_3 \\ &+\frac{li\,q_3}{q_3}\left(1+\frac{2}{5.999p^{1/2}}+\frac{q_3^2}{143.952p}+0.00001\right) \\ &+0.0038\,\frac{q_3^2}{p}+0.15328\Biggr\} \\ &-(li\,q_3/q_3-0.0263)(11.9984-3q_3^2/8p)+60.1465q_3/p, \end{aligned}$$

$$\begin{aligned} \frac{4.3731}{12} < &\frac{1}{2}\log\left(\log\frac{A}{4}\log\frac{A}{5}\right)-\log\log q_3 \\ &+\frac{li\,q_3}{q_3}(0.00014+0.00034+0.03825q_3^2/p+0.00001) \\ &+0.0263\left(1-\frac{q_3^2}{32p}\right)+0.0038q_3^2/p+\frac{60.1465}{12}\,\frac{q_3}{p}+0.15328. \end{aligned}$$

Since

$$\frac{li\, q_3}{q_3} < \frac{1}{\log q_3 - 2} < \frac{1}{4.103} < 0.24373,$$

we have

$$\begin{aligned} 0.3644 < \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) &- \log \log q_3 \\ &+ 0.24373(0.00049 + 0.03825 q_3^2/p) \\ &+ 0.0263(1 - q_3^2/32p) + 0.0038 q_3^2/p \\ &+ 5.013 q_3/p + 0.15328, \end{aligned}$$

that is,

$$\begin{aligned} 0.3644 < \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) &- \log \log q_3 + 0.0123 q_3^2/p \\ &+ 5.013 q_3/p + 0.1797. \end{aligned}$$

Then

$$\begin{aligned} \log \log q_3 < \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) &+ 0.0123 q_3^2/p \\ &+ 5.013 q_3/p - 0.1847. \end{aligned} \tag{9}$$

Since $(p/5)^{1/2} < q_3 < 3p^{1/2}$, we have

$$\begin{aligned} \log \log q_3 &< \frac{1}{2} \log \left(\log \frac{A}{4} \log \frac{A}{5} \right) + 0.0123 \times 9 \\ &\quad + \frac{5.013 \times 3}{10^3} - 0.1847 \\ &< \frac{1}{2} \log \left(\log \frac{A}{4} \right)^2, \end{aligned}$$

that is,

$$q_3 < (3/2) p^{1/2}. \tag{10}$$

For $p < 10^8$, we have by (10),

$$\begin{aligned} \frac{d}{dq_3}\left(\log \log q_3 - \frac{0.0123}{p} q_3^2 - \frac{5.013 q_3}{p} \right) &= \frac{1}{q_3 \log q_3} - \frac{0.0246 q_3}{p} - \frac{5.013}{p} \\ &> \frac{2}{3p^{1/2} \log ((3/2) \times 10^4)} - \frac{0.0246}{p^{1/2}} \frac{3}{2} - \frac{5.013}{p} > 0. \end{aligned}$$

Therefore (9) gives

$$\log\log\left(\frac{p}{5}\right)^{1/2} < \frac{1}{2}\log\left(\log\frac{3}{2}\,p^{1/2}\log\frac{6}{5}\,p^{1/2}\right)$$

$$+\frac{0.0123}{p}\left(\left(\frac{p}{5}\right)^{1/2}\right)^2+\frac{5.013}{p}\left(\frac{p}{5}\right)^{1/2} - 0.1847$$

$$< \frac{1}{2}\log\left(\log\frac{3}{2}\,p^{1/2}\log\frac{6}{5}\,p^{1/2}\right)$$

$$+ 0.00246 + \frac{5.013}{10^3(5)^{1/2}} - 0.1847$$

$$< \frac{1}{2}\log\left(\log\frac{3}{2}\,p^{1/2}\log\frac{6}{5}\,p^{1/2}\right) - 0.1799$$

or

$$0.35 < 0.1799 \times 2 < \log\frac{\log(3p^{1/2}/2)\log(6p^{1/2}/5)}{(\log(p/5)^{1/2})^2}$$

$$< \log\frac{\log(3/2\times 10^3)\log(6/5\times 10^3)}{(\log 10^3/5^{1/2})^2}$$

$$< \log 1.393 < 0.332,$$

which is absurd.

Thus the lemma is established.

7. Proof of the theorem for E. A.

THEOREM 4. *For a prime* $p\equiv 17 \pmod{24}$, *there is no E.A. in* $R(p^{1/2})$ *except possibly for* $p=17$, 41, 89, 113 *and* 137.

Proof. (1) Suppose $q_2=5$. Since

$$40q_3 < 7.5\times 40p^{1/2} = 300p^{1/2} < p$$

if $p>300^2=90000$, the theorem is true for $p>90000$. For $p<90000$, it is verified directly that the condition

$$40q_3 < p$$

holds except for $p=17$, 113, 233 and 257. But

$$233 = 2\cdot 5\cdot q_3 + 41.3 \qquad (q_3 = 11),$$

$$257 = 5q_3 + 74.3 \qquad (q_3 = 7),$$

by Lemma 15 (of I) the field $R(p^{1/2})$ has no E.A. except possibly for $p=17$, 113 and 137.

(2) Suppose $q_2\neq 5$. For $p>10^6$, the theorem follows from Lemma 32 and Lemma 16 (of I). For $p\leq 10^6$, it is verified directly that the condition

$$5q_2q_3 < p$$

is satisfied except for $p=41$, 89 and 271. But

$$271 = 3q_3 + q_2\cdot 34 \qquad (q_2 = 7,\ q_3 = 11),$$

by Lemma 14 (of I) we have the theorem.

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